Stochastic Calculus, Summer 2023, June 14, Lecture 4 Construction of the Ito Integral

Reading for this lecture (for references see the end of the lecture):

- [1] pp. 125-137
- [2] pp. 33-67
- [3] pp. 128-148
- [4] pp. 21-42

Motivation. Consider an asset whose price per share is equal to $X_t, t \geq 0$ and consider a portfolio that initially consists of Δ_0 shares. Consider the following trading strategy. Keep initial position Δ_0 up to time $t_1 \geq t_0 = 0$ and then re-balance the portfolio by taking position Δ_1 in the asset. Keep it up to time $t_2 \geq t_1$ and then re-balance the portfolio again by taking position Δ_2 is the asset. In general, we re-balance the portfolio at trading date t_i by taking position Δ_i in the asset and keeping it till the next trading date t_{i+1}. What is the the profit $I_T(\Delta)$ of the above trading strategy at time T? Clearly

$$I_T(\Delta) = \Delta_0(X(t_1)-X(t_0))+\Delta_1(X(t_2)-X(t_1))+\cdots+\Delta_{n-1}(X(t_n)-X(t_{n-1})) \quad (1)$$

and by analogy with the Reimann integral we write symbolically

$$I_T(\Delta) = \int_0^T \Delta(t)dX(t), \quad (2)$$

where $\Delta(t)$ is a piecewise constant function which is equal to Δ_i on $[t_i, t_{i+1}]$.

Construction of the stochastic integral. We fix an interval $[S,T]$ and try to make sense of

$$\int_S^T f(t,w)dX_t(w), \quad (3)$$

where $f(t,w)$ is a random function and $dX_t(w)$ refers to the increments of stochastic process X_t. Before we proceed we have to clarify a few things.

- First, we restrict attention to such functions f that for any fixed t random variable $f(t,w)$ is $\mathcal{F}_t$-measurable. To explain this restriction let's come back to our canonical example (2). Position Δ_i we take in the asset at time $t_i, i \geq 1$, may depend on the price history of the asset, $\mathcal{F}_t$, but it must be independent of the future behavior of the process X_t.
- Second, we restrict our consideration to the case when X_t is a Brownian motion. The case of general stochastic process X_t is quite similar.

The problem we face when trying to assign meaning to integral (3) is that Brownian motion paths cannot be differentiated with respect to time. If $X(t)$ is a differentiable function, then we can define

$$\int_S^T f(t,w)dX(t) = \int_S^T f(t,w)X'(t)dt, \quad (4)$$

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where the right-hand side is an ordinary integral with respect to time. That is for every trajectory of $X(t)$ we can define $\int\limits_S^T f(t,w)dX(t)$. This approach does not work for Brownian motion as we proved that the trajectories of Brownian motion are not differentiable.

Just like in the definition of the usual Riemann integral $\int\limits_S^T f(t)dt$, where $f(t)$ is a deterministic function, we start with a definition for a simple class of functions f and then extend by some approximation procedure.

Assume that $\Pi = \{t_0, t_1, \dots, t_n\}$ is a partition of $[S,T]$, i.e.

$$S = t_0 \leq t_1 \leq \cdots \leq t_n = T, \tag{5}$$

and that $f(t,w)$ is constant in t on each subinterval $[t_j, t_{j+1})$. Such a process $f(t,w)$ is called a *simple process.* We start with defining integral (3) for simple processes. Consider interval $[t_0, t_1]$. On this interval $f(t,\omega) = e_1(\omega)$ is a random quantity, but independent of t and thus it is natural to assume that

$$\int_{t_0}^{t_1} f(t,\omega)\, dB_t(\omega) = \int_{t_0}^{t_1} e_1(\omega)\, dB_t(\omega) = e_1(\omega)\left(B(t_1) - B(t_0)\right).$$

Applying this procedure to intervals $[t_2, t_1], [t_3, t_2], \dots$ we get

$$\int_S^T f(t,w)\, dB_t(w) = e_1(w)\left(B(t_1) - B(t_0)\right) \\ + e_2(w)\left(B(t_2) - B(t_1)\right) + e_3(w)\left(B(t_3) - B(t_2)\right) + \dots \tag{6}$$

Naturally, to define stochastic integral (3) for general process $f(t,w)$ we approximate it with simple processes similarly to approximation of continuous functions by stepwise constant functions in the theory of Riemann integration. But without any further assumption on approximating functions $e_i(w)$, our definition of the integral leads to difficulties. Here is an example of what kind of difficulties we can expect. Consider

$$\int\limits_0^T B_t dB_t. \tag{7}$$

Riemann integral is a limit of Riemann sums:

$$\int\limits_S^T f(t)dt \approx \sum_{i=0} f(t_i^*)(t_{i+1} - t_i), \tag{8}$$

where t_i^* is ANY point on the interval $[t_i, t_{i+1}]$. When the length of the longest interval in the partition tends to zero the limit is $\int\limits_S^T f(t)dt$. Let us point out that it was not important what point t_i^* we took inside the interval $[t_i, t_{i+1}]$. For example, it could be t_i(left point approximation) or t_{i+1}(right point approximation). Let us try to do the same for integral (7).

Left point approximation:

$$I_1 \cong \sum_i B(t_i)\left(B(t_{i+1}) - B(t_i)\right). \tag{9}$$

Right point approximation:

$$I_2 \cong \sum_i B(t_{i+1})\left(B(t_{i+1}) - B(t_i)\right). \tag{10}$$

From the independence of increments of Brownian motion and the fact that $\mathbb{E}\left[\left(B(t_{i+1}) - B(t_i)\right)\right] = \mathbb{E}\left[B(t_1)\right] = 0$ we have

$$\begin{aligned}
\mathbb{E}(I_1) &= \sum_i \mathbb{E}\left[B(t_i)\left(B(t_{i+1}) - B(t_i)\right)\right] \\
&= \sum_i \mathbb{E}\left[B(t_i)\right]\mathbb{E}\left[\left(B(t_{i+1}) - B(t_i)\right)\right] = 0. \\
\mathbb{E}(I_2) &= \sum_i \mathbb{E}\left[B(t_{i+1})\left(B(t_{i+1}) - B(t_i)\right)\right] \\
&= \sum_i \mathbb{E}\left[B(t_{i+1})^2 - B(t_{i+1})B(t_i)\right] \\
&= \sum_i \left[t_{i+1} - t_i\right] = T, \qquad (11)
\end{aligned}$$

since $\mathbb{E}\left[B(t_{i+1})^2\right] = t_{i+1}$ and $\mathbb{E}\left[B(t_{i+1})B(t_i)\right] = t_i$ as follows from

$$\mathbb{E}\left(B(t)B(s)\right) = \min(s,t). \tag{12}$$

Thus we see that depending on the choice of the point t_i^* in the approximation we can get very different results. Function $f(t,w)$ is $\mathcal{F}_t$-measurable and thus it is reasonable to choose the approximating simple function to be $\mathcal{F}_t$-measurable as well. We therefore have to choose the left end point approximation. In what follows we choose

$$t_i^* = t_i \text{ (left end point approximation)} \tag{13}$$

which leads to the Itô integral.

Remark 1. If for each $t \geq 0$ random variable $f(t,w)$ is $\mathcal{F}_t$ measurable we say that the process $f(t,w)$ is $\mathcal{F}_t$-adapted. For example, if $\mathcal{F}_t$ is filtration of Brownian motion then the process $f_t(t,w) = B(t/2)$ is $\mathcal{F}_t$-adapted, while $f_t(t,w) = B(2t)$ is not.

Properties of the Itô integral for simple processes. The Itô integral (3) is defined as the gain from treading in the martingale B_t. A martingale has no tendency to rise or fall and hence it is to be expected that

$$I_t(f) = \int_0^t f(t,w)dB_t \tag{14}$$

also has no tendency to rise or fall.

Theorem 2. *Itô integral is a martingale.*

Proof. see [1], pages 128-129. □

Because $I_t(f)$ is a martingale and $I_0 = 0$ we have $\mathbb{E}I_t(f) = 0$ for all $t \geq 0$. It follows that $\mathrm{Var}I_t(f) = \mathbb{E}I_t^2(f)$ can be evaluated by the formula in the following theorem.

Theorem 3. *The Itô integral satisfies*

$$\mathbb{E}I_t^2(f) = \mathbb{E}\int\limits_0^t f(s,w)^2 ds \tag{15}$$

Proof. For the simplicity of notation we introduce $\Delta B_i = B(t_{i+1}) - B_t(t_i), e_i = e_i(w)$. Then by definition

$$I_t(f) = \int_0^t f(s,\omega)\, dB_s = \sum_i e_i \Delta B_i \tag{16}$$

and

$$\left(\int_S^T f(t,w)\, dB_t\right)^2 = \left(\sum_i e_i \Delta B_i\right)^2 = \sum_{i,j} e_i e_j \Delta B_i \Delta B_j. \tag{17}$$

Taking expectation

$$\mathbb{E}\left(\int_S^T f(t,\omega)\, dB_t\right)^2 = \sum_{i,j} \mathbb{E}\left(e_i e_j \Delta B_i \Delta B_j\right). \tag{18}$$

$$\mathbb{E}\left(e_i e_j \Delta B_i \Delta B_j\right) = \begin{cases} 0 = \mathbb{E}\left(\Delta B_j\right), \text{ if } i < j \\ \mathbb{E}(e_i^2 \Delta B_i^2), \text{ if } i = j \end{cases}$$

For $i = j$ we use independence of increments property to conclude

$$\mathbb{E}\left(e_i^2 \left(\Delta B_i\right)^2\right) = \mathbb{E}\left(e_i^2\right)\mathbb{E}\left(\Delta B_i\right)^2 = \mathbb{E}\left(e_i^2\right)\left(t_{i+1} - t_i\right) = \mathbb{E}\left(e_i^2\right)\Delta t_i. \tag{19}$$

$$\sum_i \mathbb{E}\left(e_i^2\right)\Delta t_i = \mathbb{E}\sum_i \left(e_i^2\right)\Delta t_i = \mathbb{E}\int_0^t \phi\left(t,\omega\right)^2 dt \tag{20}$$

□

Theorem 4. *Quadratic variation of the stochastic integral* (3) *is equal to*

$$\int\limits_0^T f^2(t,w)dt = \sum_i e_i^2 \Delta t_i. \tag{21}$$

For this purpose we consider the quantity

$$\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2 \tag{22}$$

and prove that it approaches 0 as $||\Pi|| \to 0$. We first rewrite it as

$$\begin{aligned} \mathbb{E}\Big(\sum_i e_i^2 (\Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\Big)^2 &= \mathbb{E}\left(\sum_i e_i^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big]\right)^2 \\ &= \mathbb{E}\sum_{i,j} e_i^2 e_j^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big]\Big[(\Delta B_j)^2 - \Delta t_j\Big]. \end{aligned} \tag{23}$$

Just as in the calculation of the quadratic variation of the Brownian motion let us split the above sum in two sums: in the first one we keep the terms with $i \neq j$ and in the second one we keep terms with $i = j$.

Let us first look at terms with $i \neq j$, for instance $i < j$. Then $\Big[(\Delta B_j)^2 - \Delta t_j\Big]$ is independent of $e_i^2 e_j^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big]$ an thus

$$\mathbb{E}e_i^2 e_j^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big]\Big[(\Delta B_j)^2 - \Delta t_j\Big] = \mathbb{E}e_i^2 e_j^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big]\mathbb{E}\Big[(\Delta B_j)^2 - \Delta t_j\Big]$$
$$= \mathbb{E}e_i^2 e_j^2 \Big[(\Delta B_i)^2 - \Delta t_i\Big] \cdot 0 = 0. \tag{24}$$

Let us now consider the case of $i = j$. Then

$$\mathbb{E}e_i^4 \Big[(\Delta B_i)^2 - \Delta t_i\Big]^2 = \mathbb{E}e_i^4 \mathbb{E}\Big[(\Delta B_i)^2 - \Delta t_i\Big]^2, \tag{25}$$

since random variables e_i^4 and $\Big[(\Delta B_i)^2 - \Delta t_i\Big]^2$ are independent. It follows from the fact that e_i is $\mathcal{F}_t$-measurable and thus ΔB_i is independent of e_i. But $\mathbb{E}\Big[(\Delta B_i)^2 - \Delta t_i\Big]^2 = 2(\Delta t_i)^2$ and thus

$$\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2 = \sum_i 2\mathbb{E}e_i^4 (\Delta t_i)^2$$
$$\leq ||\Pi|| \sum_i 2\mathbb{E}e_i^4 \Delta t_i. \tag{26}$$

Since $\sum_i 2\mathbb{E}e_i^4 \Delta t_i$ converges to $\int_0^T \mathbb{E}f^4 dt < \infty$. Thus

$$\mathbb{E}\left(\sum_i (e_i \Delta B_i)^2 - \sum_i e_i^2 \Delta t_i\right)^2 \to 0 \tag{27}$$

and quadratic variation of $I(f)$ is proved to be

$$\int_0^T f^2(t, w) dt. \tag{28}$$

Itô integral for general functions. We now describe the class of functions $f(t, w)$ for which the Itô integral will be defined.

Definition 5. *Let $V = V(S, T)$ be the class of functions $f(t, w) : [0, \infty) \times \Omega \to \mathbb{R}$, such that:*

(1) *$f(t, w)$ is $\mathcal{F}_t$ – adapted*
(2) $\int_S^T f(t, w)^2 \, dt < \infty$.

We claim that each function $f \in V(S, T)$ can be approximated by a sequence $\{\varphi_n\}_{n=1,2,\ldots}$ of simple functions (or equivalently, by a sequence of simple processes) in the sense that as $n \to \infty$

$$\mathbb{E}\int_S^T (f - \varphi_n)^2 \to 0. \tag{29}$$

The approximation is done is three steps:
Step 1 (Approximate bounded continuous functions with simple functions)

Let $g \in V$ be bounded, i. e., every trajectory $g(.,w)$ (w is fixed and t changes) is continuous. Then, there exists a sequence of simple functions $\varphi_n \in V$, such that as $n \to \infty$

$$\mathbb{E}\int_S^T (g - \varphi_n)^2\, dt \to 0. \tag{30}$$

Step 2 (Approximate bounded functions with bounded continuous functions) Let $h \in V$ be bounded, then there exists a sequence of bounded continuous functions g_n, such that

$$\mathbb{E}\int_S^T (h - g_n)^2\, dt \to 0. \tag{31}$$

Step 3 (Approximate general functions with bounded functions) Let $f \in V$, then there exists a sequence of bounded functions h_n, such that

$$\mathbb{E}\int_S^T (f - h_n)^2\, dt \to 0. \tag{32}$$

Putting together steps 1,2 and 3 we get that for any function $f(t,w) \in V$ there exists a sequence of simple functions $\varphi_n(t,w)$ such that (30) is true. We define then the Itô integral of function $f(t,w)$ as

$$I(f) = \int_S^T f(t,\omega)\, dB_t = \lim_{n\to\infty} I(\varphi_n). \tag{33}$$

Question: Why does the limit exist and in what sense?

Answer: By Theorem 3 we have that

$$\begin{aligned}\mathbb{E}(I(\varphi_n) - I(\varphi_m))^2 &= \mathbb{E}\int_S^T (\varphi_n - \varphi_m)^2 dt \\ &\le \mathbb{E}\int_S^T (f - \varphi_m)^2 dt + \mathbb{E}\int_S^T (\varphi_n - f)^2 dt \to 0. \end{aligned}\tag{34}$$

Thus the sequence of random variables $\left\{\int_S^T \varphi_n(t,\omega)\, dB_t\right\}$ forms a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Since $L_2(\Omega, \mathcal{F}, \mathbb{P})$ is a complete space then there exists a limit of $I(\varphi_n)$ as an element of $L_2(\Omega, \mathcal{F}, \mathbb{P})$. This limit is by definition the Itô integral $I(f)$.

Example: Compute $\int_0^T B_t dB_t$.

By definition

$$\int_0^T B_t dB_t = \lim_{n\to\infty}\int_0^T \varphi_n(t,\omega)\, dB_t, \tag{35}$$

where φ_n is such that $\mathbb{E}\int_0^T (\phi_n - B_t)^2\, dt \to 0$ and φ_n is $\mathcal{F}_t$-adapted.

As we already saw in the beginning of the lecture we can approximate $f(t,w) = B_t$ by partitioning $[0,T]$ into $[0,t_1], [t_1,t_2], \ldots, [t_n{-}1, t_n = T]$ and defining $\varphi_n(t,w) = B(t_i)$ for $t \in [t_i, t_{i+1}]$. Let us first check that φ_n indeed approximated f in the sense of (29):

$$\mathbb{E}\int_0^T (\phi_n - B_t)^2\,dt = \mathbb{E}\sum_i \int_{t_i}^{t_{i+1}} (\phi_n - B_t)^2\,dt = \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E}\left(B\left(t_i\right) - B\left(t\right)\right)^2 dt$$
$$= \sum_i \int_{t_i}^{t_{i+1}} (t - t_i)\,dt = \sum_i \frac{(t_{i+1} - t_i)^2}{2}. \tag{36}$$

If we define $\max(t_{i+1} - t_i) = M_n$ then

$$\sum_i \frac{(t_{i+1} - t_i)^2}{2} \le \sum_i \frac{t_{i+1} - t_i}{2} M_n = \frac{M_n}{2} \sum_i (t_{i+1} - t_i) = \frac{M_n}{2} T \to 0. \tag{37}$$

Thus we have to compute $\int_0^T \phi_n dB_t = \sum_i B\left(t_i\right)\Delta B_i$, where $\Delta B_i = B(t_{i+1}) - B(t_i)$.

We use the following identity

$$\Delta B_i^2 = B(t_{i+1})^2 - B(t_i)^2 = (B(t_{i+1}) - B(t_i))^2 + 2B(t_{i+1})B(t_i) - 2B(t_i)^2$$
$$= (B(t_{i+1}) - B(t_i))^2 + 2B(t_i)(B(t_{i+1}) - B(t_i)). \tag{38}$$

Summing both parts over i we get

$$B_T^2 = \sum_i (B(t_{i+1}) - B(t_i))^2 + 2I\left(\phi_n\right). \tag{39}$$

Therefore

$$I\left(\varphi_n\right) = \frac{B_T^2}{2} - \frac{1}{2}\sum_i \left(B_{i+1} - B_i\right)^2, \text{ but } \sum_i \left(B_{i+1} - B_i\right)^2 \underset{L_2}{\longrightarrow} T. \tag{40}$$

Finally,

$$\int_0^T B_t dB_t = \frac{B_T^2}{2} - \frac{T}{2}. \tag{41}$$

References

[1] Steven Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*
[2] Richard Durrett, *Stochastic Calculus: A Practical Introduction.*
[3] Ioannis Karatzas, Steven E. Shreve, *Brownian Motion and Stochastic Calculus.*
[4] Bernt Oksendal, Stochastic Differential Equations