Reading for this lecture:

(1) [1] pp. 108-115
(2) [2] pp. 25-31
(3) [3] pp. 95-97

Background: Vanilla Barrier Options. In finance, a barrier option is a type of contract where option to exercise at maturity depends on some observable (also called underlying of the option) crossing or reaching a given level, the so-called barrier. There are several types of vanilla barrier options ("vanilla" stands for simple and liquid on the market instrument). Some "knock-out" when the underlying asset price crosses a barrier (i.e., they become worthless). If the underlying asset price begins below the barrier and must cross the barrier above it to cause the knock-out, the option is said to be *up-and-out.* A *down-and-out* option has the barrier below the initial asset price and knocks out if the asset price falls below the barrier. Other options "knock-in" at a barrier (i.e., payoff zero unless they cross a barrier). Knock-in options also fall in two categories: *up-and-in* and *down-and-in.* The payoff at expiration is often a fixed amount, a call or a put. There are also more complicated barrier options, for instance, range accrual options which for a specified financial observable, for instance 3-month Libor rate or 2-year CMS rate, pays a fixed amount multiplied by the fraction of observations when the index is inside a specified range.

Later in this lecture we treat the simplest case. We price an up-and-out option whose payoff at expiration is a call and we assume that the stock price is modelled by Brownian motion. Assumption that the stock price can be modelled by Brownian motion is rather unrealistic as Brownian motion (even with large drift) can be negative with positive probability. Later in the course we will be able to apply similar calculations for the more realistic case when the asset price follows a geometric Brownian motion.

Running maximum and first passage time. For a stochastic process $X_t, t \geq 0$ we define the running maximum as

$$M_t = \max_{0\leq s\leq t} X_s. \tag{1}$$

For this process to be well defined we assume that the process X_t is has continuous trajectories(in fact, we need only with probability one). Closely related to running maximum is the first passage time which is defined as

$$T_a = \inf\{t > 0 : X_t = a\}, \tag{2}$$

that is for a fixed a time T_a is the first time when X_t reaches level a. From the definitions it is clear that events $\{T_a < t\}$ and $\{M_t \geq a\}$ coincide, i.e.,

$$w \in \{T_a < t\} \Longleftrightarrow w \in \{M_t > a\}. \tag{3}$$

From the examples in the beginning of the lecture one can conclude that the value of many barrier options depends on the behavior of the maximum asset price prior to option expiration, or equivalently, on the distribution of the running maximum. For instance, the knowledge of the distribution of the running maximum (or equivalently of the first passage time) is enough to compute the value of the simple knock-in/out options.

But it turns out that for a general stochastic process X_t the distribution of the running maximum is rather complicated and often cannot be written out in a closed form.

Brownian motion. In the case when stochastic process X_t is a Brownian motion we can explicitly calculate the distribution of the running maximum and hitting time.

Theorem 1 (Reflection Principle.)**.** *Let* $a > 0$*. Then* $\mathbb{P}(T_a < t) = 2\mathbb{P}(B_t > a)$.

Remark 2. Before the start of the proof let us rewrite the above equation as

$$\mathbb{P}(T_a < t) = 2\int_a^\infty \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}dx. \tag{4}$$

To find the probability density of T_a we change variables $x = \frac{\sqrt{t}a}{\sqrt{s}}$. Then $dx = -\frac{\sqrt{t}a}{2s^{3/2}}$ and

$$\mathbb{P}(T_a < t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}}e^{-a^2/2s}ds, \tag{5}$$

and thus the density of T_a is $\frac{a}{\sqrt{2\pi s^3}}e^{-a^2/2s}$.

Proof. Assume that B_t hits level a at some time $s < t$. From the independence of increments property it follows that $B_t - B_{T_a}$ is independent of what happened before time T_a. Moreover, the increment $B_t - B_{T_a}$ is normally distributed. Since normal distribution is symmetric and probability of B_t being equal to a is zero we obtain

$$\mathbb{P}(T_a < t, B_t > a) = \frac{1}{2}\mathbb{P}(T_a < t). \tag{6}$$

We multiply by 2 and notice that event $\{B_t > a\}$ is a subset of the event $\{T_a < t\}$, thus

$$\mathbb{P}(T_a < t) = 2\mathbb{P}(T_a < t, B_t > a) = 2\mathbb{P}(B_t > a). \tag{7}$$

□

Remark 3. Truly speaking, the fact that $B_t - B_{T_a}$ is independent of $\mathcal{F}_{T_a}$ ("information" available at time T_a) needs to be proved using the strong Markov property for Brownian motion. We refer to [2] for the proof.

One can go further and generalize the result of Theorem 1. Let $u < v \leq a$, then using the reflection principle we can easily show that

$$\mathbb{P}(T_a < t, u < B_t < v) = \mathbb{P}(2a - v < B_t < 2a - u). \tag{8}$$

Since events $\{T_a < t\}$ and $\{M_t > a\}$ are equivalent we obtain

$$\mathbb{P}(M_t > a, u < B_t < v) = \mathbb{P}(2a - v < B_t < 2a - u). \tag{9}$$

Now let the interval (u, v) shrink to x, so that

$$\mathbb{P}(M_t > a, B_t = x) = \mathbb{P}(B_t = 2a - x) = \frac{1}{\sqrt{2\pi t}}e^{-(2a-x)^2/(2t)}. \tag{10}$$

Differentiating with respect to a we get the joint density

$$\mathbb{P}(M_t = a, B_t = x) = \frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}. \tag{11}$$

Knock-In/Out Option on Brownian motion. Let us consider the option which is knocked-out if before the expiration time t asset price crosses barrier M. The payoff at expiration is a call with strike K. Let us assume for simplicity that the interest rate is equal to zero so that the value of the option is equal to

$$\mathbb{E}\,\mathbf{1}_{M_t<M}(B_t-K)_+. \tag{12}$$

In order to calculate the above expectation we use our knowledge of the joint density of M_t and B_t given by (11):

$$\begin{aligned}
\mathbb{E}\,\mathbf{1}_{M_t<M}(B_t-K)_+ &= \iint \mathbf{1}_{a<M}(x-K)_+\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}dadx \\
&= \int_K^M\int_x^M (x-K)\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}dadx \\
&= \int_K^M (x-K)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}dx - \int_K^M (x-K)\frac{e^{-(x-2M)^2/2t}}{\sqrt{2\pi t}}dx. \qquad (13)
\end{aligned}$$

The above difference could be easily expressed in terms of the cumulative normal distribution. Indeed, the first term is a difference of the following two integrals

$$\begin{aligned}
&\int_K^M x\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}dx = \sqrt{t}\int_{K/\sqrt{t}}^{M/\sqrt{t}} x\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \sqrt{\frac{t}{2\pi}}\left(e^{-K^2/2t}-e^{-M^2/2t}\right) \\
&\int_K^M K\frac{e^{-x^2/2t}}{\sqrt{2\pi t}} = K\int_{K/\sqrt{t}}^{M/\sqrt{t}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = K\left(N\Big(\frac{M}{\sqrt{t}}\Big)-N\Big(\frac{K}{\sqrt{t}}\Big)\right), \qquad (14)
\end{aligned}$$

where $N(x) = \int\limits_{-\infty}^{x} e^{-z^2/2}/\sqrt{2\pi}dz$ is the cumulative normal distribution.

Let us remark that the same method applies for the case of general payoff and knock-out condition. If the payoff at expiration is given by function $f(x)$ and knock-out condition is given by $M_t \in A$ then the value of the option is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{1}_{a\in A}f(x)\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}dadx. \tag{15}$$

One has to compute the above integral numerically, for instance, by using 2D Simpson's rule.

Let us say a few words why having a closed form solution for vanilla options is so important. The reason is that the greeks of the closed form solution could be calculated analytically while otherwise the greeks are computed numerically. Calculating derivatives numerically could be very noisy and may lead to unstable hedges which is quite undesirable taking into consideration the transaction cost.

The first passage time. Finally, let us compute the distribution of the first passage time using approach different from the presented above. In particular, we use the martingale approach.

Let us start with the proof that for any positive constant σ stochastic process $Z_t = e^{\sigma B_t - \sigma^2 t/2}$ is a martingale. Indeed, for $s < t$

$$\begin{aligned}\mathbb{E}\left(e^{\sigma B_t - \sigma^2 t/2} | \mathcal{F}_s\right) &= \mathbb{E}\left(e^{\sigma(B_t - B_s) + \sigma B_s - \sigma^2 t/2} | \mathcal{F}_s\right) \\ &= e^{\sigma B_s - \sigma^2 t/2}\mathbb{E}\left(e^{\sigma(B_t - B_s)} | \mathcal{F}_s\right) \qquad (16) \\ &= e^{\sigma B_s - \sigma^2 t/2} e^{\sigma^2 (t-s)/2}. \qquad (17)\end{aligned}$$

Equation (16) follows from the fact that B_s is measurable in σ-algebra $\mathcal{F}_s$, or in other words, with information up to time s at your disposal you know B_s. Equation (17) follows from the independence of increments property and the fact that for a normally distributed random variable $X = N(a, \sigma^2)$

$$\mathbb{E}e^{uX} = e^{ua + \frac{1}{2}u^2\sigma^2}. \qquad (18)$$

Let us use the fact that Z_t is a martingale to find the moment generating function of the first passage time T_a. Let us apply the martingale property at moment T_a :

$$\mathbb{E}\left(Z_{T_a} | \mathcal{F}_0\right) = Z_0, \qquad (19)$$

or equivalently

$$\mathbb{E}e^{\sigma B_{T_a} - \sigma^2 T_a/2} = 1. \qquad (20)$$

But $B_{T_a} = a$ by the definition of the first passage time. Thus

$$\mathbb{E}e^{\sigma a - \sigma^2 T_a/2} = 1 \Longleftrightarrow \mathbb{E}e^{-\sigma^2 T_a/2} = e^{-\sigma a}. \qquad (21)$$

Changing variable $\sigma^2 = 2u$

$$\mathbb{E}e^{-uT_a} = e^{-a\sqrt{2u}}. \qquad (22)$$

Thus we obtained the moment generating function of the first passage time T_a. Why is it important? Because by the moment generating function we can unambiguously recover the distribution function of T_a.

References

[1] Steven Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*
[2] Richard Durrett, *Stochastic Calculus: A Practical Introduction.*
[3] Ioannis Karatzas, Steven E. Shreve, *Brownian Motion and Stochastic Calculus.*