Connections between Orthogonal Polynomials, Three-Term Recurrences, Generating Function and the Governing ODE: The Legendre Case

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April 2023

1 Introduction

In this paper, we will study the Three-Term Recurrence, Generating Function, and Governing ODE of the Legendre Polynomials in the first section, where we will finally derive that these properties are actually equivalent. While in the second section, we will explore the wide applications of Legendre Polynomials.

2 Proving Three-Term Recurrence of Legendre Polynomials Using Gram-Schmidt

In this section, we first assume we know the Governing ODEs of Legendre Polynomials, which is the Legendre Equation. In class, using Sturm-Liouville Theory, we've seen that Legendre Polynomials are orthogonal, we now use the Gram Schmidt process to get P_{n+1}:

$$p_{n+1}(x) = a'_n(x^{n+1} - \sum_{k=0}^{n} \frac{(p_k(x), x^n)}{(p_k(x), p_k(x)} p_k(x))$$

where a'_n is the scaling constant. Furthermore, notice that: xp_n is also a degree $n+1$ polynomial, instead of using the equation above, we can apply the orthogonality and create:

$$p_{n+1}(x) = a_n(xp_n - \sum_{k=0}^{n} \frac{(p_k(x), xp_n(x))}{(p_k(x), p_k(x)} p_k(x))$$

In this form, because p_k is orthogonal to p_j for all $j < k$, therefore the inner product (xp_k, p_{n-1}) is zero as long as $k+1 < n$.
So we have

$$\begin{aligned} p_{n+1}(x) &= a_n(xp_n(x) - \frac{(p_n(x), xp_n(x))}{(p_n(x), p_n(x))} p_n(x) - \frac{(p_n(x), xp_{n-1}(x))}{(p_{n-1}(x), p_n(x))} p_{n-1}(x)) \\ &= a_n((x - \frac{(p_n(x), xp_n(x))}{(p_n(x), p_n(x))})p_n(x) - \frac{(p_n(x), xp_{n-1}(x))}{(p_{n-1}(x), p_{n-1}(x))} p_{n-1}(x)) \end{aligned}$$

It gives us the three term recurrence of the Legnedre polynomials by orthogonality and inner product of the Legendre polynomials.

We can also find the formula to define p_{n+1} by p_n, p_{n-1}, and n only.
From the previous part, $p_{n+1}(x) = (a_n x + b_n)p_n(x) + c_n p_{n-1}(x)$.

Since we have $p_n(1) = 1$ and $p_n(-1) = (-1)^n$, we have $a_n + b_n + c_n = 1$ and $a_n - b_n + c_n = 1$ for odd n. This tells us that $c_n = 1 - a_n$, and $b_n = 0$ for all terms of Legendre polynomials. Then, to compute the inner porduct explicitly, we first need to evaluate $(p_n(x), p_n(x))$ in terms of n.
Using the Rodrigues' formula [1]:

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Using IVP for n times we get Then

$$I_n = \int_{-1}^{1} p_n(x)p_n(x)dx = \frac{(-1)^n(2n)!}{4^n(n!)^2}\int_{-1}^{1}(x^2-1)^n dx$$

where

$$\int_{-1}^{1}(x^2-1)^n dx$$

can also be solved by integration by parts repeated n times:

$$\int_{-1}^{1}(x^2-1)^n dx = (-1)^n \frac{n!^2}{(2n)!}\frac{2^{2n+1}}{2n+1}$$

Therefore

$$I_n = \frac{2}{2n+1}$$

Compute $(p_n(x), xp_n(x))$, it turns out to be zero by IBP.(Coincide with our conclusion on b_n before)
Similarly using Rodrigues' formula and IBP we have:

$$(p_n(x), xp_{n-1}(x)) = \frac{2n}{(2n-1)(2n+1)}$$

Then the formula becomes:

$$p_{n+1}(x) = a_n x p_n(x) + c_n p_{n-1}(x)$$

So we have 2 unknowns a_n, c_n and 2 equations $a_n + c_n = 1, c_n = -\frac{(p_n(x), xp_{n-1}(x))}{(p_{n-1}(x), p_{n-1}(x))}a_n$. Thus, solving the system and the exact recurrence formula for Legendre polynomial is:

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_n(x) = \frac{2n+1}{n+1}xp_n(x) - \frac{n}{n+1}p_{n-1}(x), n > 2$$

This formula gives us the explicit recurrence relationship between the 3 term Legendre polynomials, which only depends on the degree n.

3 Proving Three-Term Recurrence of Legendre Polynomials Using Generating Functions

In this section, we first assume we know the generating function of Legendre Polynomials to derive the 3-term recurrence between them, we shall discuss the reverse argument in the next section. We will use

$$g(t, x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

we first take the partial derivative of g regarding t, we can see that

$$\frac{\partial g(t,x)}{\partial t} = \int_{n=0}^{\infty} nP_n(x)t^{n-2} = \frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}}$$

we multiply $1-2xt+t^2$, and use generating function we get

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

we could simplify it as

$$[P_1(x) - xP_0(x)]t^0 + \sum_{n=1}^{\infty}[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)]t^n = 0$$

since left hand side must vanish for all t we get

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{1}$$

thus it gives another three term recurrence proof other than through Gram-Schmidt.

4 Proving Generating Function Using Three-Term Recurrence

In this section, we shall prove the generating function of Legendre Polynomial using the three-term recurrences. For fixed t, we write the generating function of $P_n(x)$ as:

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n.$$

By taking the derivative, we have:

$$\frac{dG(x,t)}{dt} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}.$$

We now multiply t^{n+1} on both sides of equation (1), and sum it from $n=1$ to ∞:

$$\sum_{n=1}^{\infty}(n+1)P_{n+1}(x)t^{n+1} = \sum_{n=1}^{\infty}[(2n+1)xP_n(x) - nP_{n-1}(x)]t^{n+1}.$$

$$\sum_{n=1}^{\infty}(n+1)P_{n+1}(x)t^{n+1} = 2xt\sum_{n=1}^{\infty} nP_n(x)t^n + t^2\sum_{n=1}^{\infty}[xP_n(x) - nP_{n-1}(x)]t^{n-1}.$$

Substitute $n+1$ by m, and $n-1$ by k,

$$\sum_{m=2}^{\infty} mP_m(x)t^m = 2xt\sum_{n=1}^{\infty} nP_n(x)t^n + t^2\sum_{k=0}^{\infty}[xP_{k+1}(x) - (k+1)P_k(x)]t^k.$$

We can write two of these summations as:

$$\sum_{m=2}^{\infty} mP_m(x)t^m = t\frac{dG(x,t)}{dt} - tP_1(x) = \frac{dG(x,t)}{dt} - tx$$

and

$$\sum_{k=0}^{\infty}(k+1)P_k(x)t^k = t\frac{dG(x,t)}{dt} + G(x,t)$$

Here, we will introduce a Lemma of Generating Function, the proof of it is given by [1].

Lemma 4.1 *If sequence $f_{n}{}_{n=0}^{\infty}$ has generating function $f(x)$ and w is a positive integer, then we have:*

$$\sum_{n=0}^{\infty} f_{n+w}x^n = \frac{1}{x^w}(f(x) - \sum_{i=0}^{w-1} f_i x^i)$$

Using this Lemma, we can reduce another summation:

$$x\sum_{k=0}^{\infty} xP_{k+1}(x)t^k = G(x,t) - P_0(x) = G(x,t) - 1$$

Hence we have now simplified the infinite summation equation to a differential equation:

$$x\frac{dG(x,t)}{dt} - tx = 2xt^2\frac{dG(x,t)}{dt} + xt(G(x,t)-1) - t^2(t\frac{dG(x,t)}{dt} + G(x,t))$$

which is simply:

$$\frac{dG(x,t)}{dt}(t - 2xt^2 - t^3) = G(x,t)(xt - t^2)$$

which is a simple separable function, which leads to the solution:

$$G(x,t) = \frac{1}{\sqrt{1-2tx+t^2}}$$

5 Proving Legendre's Equation using Three-Term Recurrence and Generating Function

In this section, our motivation is to derive Legendre's Equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{2}$$

we now need to take the partial derivative of g regarding x to explore another property

$$\frac{\partial g(t,x)}{\partial x} = \sum_{n=0}^{\infty} P_n'(x)t^n = \frac{t}{(1-2xt+t^2)^{\frac{3}{2}}}$$

using generating function again, we get

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x) - t\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

let t=0, we get

$$P_{n+1}'(x) + P_{n-1}'(x) = 2xP_n'(x) + P_n(x) \tag{3}$$

we take derivative of (1), we get

$$(n+1)P_{n+1}'(x) + nP_{n-1}'(x) = (2n+1)[P_n(x) + xP_n'(x)] \tag{4}$$

we use (3) and (4), we get the

$$P_{n-1}'(x) = -nP_n(x) + xP_n'(x) \tag{5}$$

then we further simplify to get

$$(1-x^2)P_n''(x) - 2xP_n'(x) + nP_n(x) + n[xP_n'(x) - P_{n-1}'(x)] = 0$$

we use (3) to simplify it into the (2), which means the Legendre polynomials defined by the generating function satisfy Legendre's Equation. We have now successfully proved the equivalence relationship between three-terms recurrence, generating functions, and governing ODE.

6 Applications of Legendre Polynomials

6.1 Legendre Polynomials in Solving Laplace's Equation

Laplace's equation appears in various fields of physics, such as electrostatics, fluid dynamics, and heat conduction. In spherical coordinates (r, θ, ϕ), Laplace's equation is given by:

$$\nabla^2 V(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

To solve Laplace's equation, we use the method of separation of variables by assuming a solution of the form:

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Substituting this into Laplace's equation and dividing by $V(r,\theta,\phi)$, we obtain:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

Separating the variables, we have:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1), \qquad \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = l(l+1), \qquad \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2,$$

where l and m are constants. The second equation is the associated Legendre equation, which has solutions in the form of associated Legendre functions:

$$\Theta(\theta) = P_l^m(\cos\theta),$$

where $P_l^m(x)$ are the associated Legendre functions, which can be expressed in terms of Legendre polynomials $P_l(x)$ when $m = 0$. Specifically:

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}(\frac{d}{dx})^m P_l(x)$$

and $P_l(x)$ is the l^{th} Legendre polynomial:

$$P_l(x) = \frac{1}{2^l l!}(\frac{d}{dx})^l(x^2-1)^l$$

Thus, Legendre polynomials play a crucial role in solving Laplace's equation in spherical coordinates.

6.2 Gaussian Quadrature

Gaussian quadrature, also known as Gauss-Legendre quadrature, is a numerical integration technique that uses Legendre polynomials to approximate definite integrals. The method involves choosing optimal nodes and weights in such a way that the resulting quadrature formula is exact for polynomials of degree $2n-1$ or lower, where n is the number of nodes.

The nodes x_i and weights w_i of the Gaussian quadrature can be determined from the roots and the corresponding weights of the Legendre polynomials. Specifically, the nodes x_i are the roots of the Legendre polynomial $P_n(x)$ of degree n:

$$P_n(x_i) = 0.$$

The weights w_i can be computed using the derivative of the Legendre polynomial:

$$w_i = \frac{2(1-x_i^2)}{(nP_{n-1}(x_i))^2}.$$

Given the nodes x_i and weights w_i, we can approximate a definite integral on the interval $[-1, 1]$ as follows:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

To approximate a definite integral on an arbitrary interval $[a, b]$, we can use a linear transformation:

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right).$$

In conclusion, Gaussian quadrature demonstrates the power and versatility of Legendre polynomials in the field of numerical integration. By utilizing the roots of Legendre polynomials as nodes and calculating the corresponding weights, this method efficiently and accurately approximates definite integrals for smooth functions that can be well represented by polynomials. The connection between Legendre polynomials and Gaussian quadrature highlights the importance of these orthogonal polynomials in practical applications, as they enable precise numerical evaluations of integrals that are essential in various scientific and engineering fields.

7 Bibliography

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